Math 132 - October 20, 2017
Solutions

7.4 Partial Fractions

Partial Fraction Steps:

1. Long Division (if deg top > deg bottom)
2. Factor the denominator
   (linear and quadratic irreducibles)
3. Determine the "form" (linear, quadratic, repeated)
4. Solve for constants
5. Integrate term by term

Warm-up Problems

1. **Clicker** Solve the partial fraction decomposition

$$\frac{x^2+x+2}{(x-2)(x-1)(x+1)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{x+1}$$

(a) $C = 1$ (b) $C = \frac{1}{3}$ **Correct** (c) $C = -2$ (d) $C = \frac{8}{3}$

Solution: You should get the system:

$$\begin{aligned} A + B + C &= 1 \\ -B - 3C &= 1 \\ -A - 2B + 2C &= 2 \end{aligned}$$

$A = \frac{8}{3}, B = -2, C = \frac{1}{3}$.

2. Integrate $\displaystyle\int \frac{x^5 - 5x^3 + x^2 + 5x + 2}{x^3 - 2x^2 - x + 2}\,dx$

Note: You have to do every step in this problem, so probably not reasonable to complete this quickly. But it would probably be worth your time to work on this problem after class.

Solution:

$$\begin{aligned} \int \frac{x^5 - 5x^3 + x^2 + 5x + 2}{x^3 - 2x^2 - x + 2}\,dx &= \int \left(x^2 + 2x + \frac{x^2+x+2}{(x-1)(x-2)(x+1)}\right) dx \\ &= \int \left(x^2 + 2x + \frac{1}{3(x+1)} - \frac{2}{x-1} + \frac{8}{3(x-2)}\right) dx \\ &= \frac{1}{3}x^3 + x^2 + \frac{1}{3}\ln|x+1| - 2\ln|x-1| + \frac{8}{3}\ln|x-2| + C \end{aligned}$$

Class Problems

Note: Problems 3 through 8 are linked (corresponding parts correspond).

3. Long division practice:

(a) $\dfrac{6x^4 + 3x^3 - 172x^2 + 102x - 329}{x^2 + x - 30}$

Solution: $\dfrac{6x^4 + 3x^3 - 172x^2 + 102x - 329}{x^2 + x - 30} = 6x^2 - 3x + 11 + \dfrac{x+1}{x^2 + x - 30}$

(b) $\dfrac{3x^5 - 2x^4 + 13x^3 - 8x^2 + 5x + 1}{x^3 + 4x}$

Solution: $\dfrac{3x^5 - 2x^4 + 13x^3 - 8x^2 + 5x + 1}{x^3 + 4x} = 3x^2 - 2x + 1 + \dfrac{x+1}{x^3 + 4x}$

(c) $\dfrac{x^8 + 4x^6 - x^5 - 4x^3 + x + 1}{x^5 + 4x^3}$

Solution: $\dfrac{x^8 + 4x^6 - x^5 - 4x^3 + x + 1}{x^5 + 4x^3} = x^3 - 1 + \dfrac{x+1}{x^5 + 4x^3}$

(d) $\frac{2x^5-4x^3+3x^2+3x+1}{x^4-2x+1}$

Solution: $\frac{2x^5-4x^3+3x^2+3x+1}{x^4-2x+1} = 2x + \frac{3x^2+x+1}{x^4-2*x+1}$

(e) $\frac{2x^6-2x^5+20x^4-20x^3+21x^2-17x+1}{x^6-x^5+10x^4-10x^3+9x^2-9x}$

Solution: $\frac{2x^6-2x^5+20x^4-20x^3+21x^2-17x+1}{x^6-x^5+10x^4-10x^3+9x^2-9x} = 2 + \frac{3x^2+x+1}{x^6-x^5+10x^4-10x^3+9x^2-9x}$

4. Factoring Practice:

(a) x^2+x-30

Solution: $x^2+x-30 = (x-5)(x+6)$

(b) x^3+4x

Solution: $x^3+4x = x(x^2+4)$

(c) x^5+4x^3

Solution: $x^5+4x^3 = x^3(x^2+4)$

(d) x^4-2x^2+1

Solution: $x^4-2x^2+1 = (x-1)^2(x+1)^2$

(e) $x^6-x^5+10x^4-10x^3+9x^2-9x$

Solution: $x^6-x^5+10x^4-10x^3+9x^2-9x = x(x-1)(x^2+9)(x^2+1)$

5. Writing down partial fraction form

(a) $\frac{x+1}{(x-5)(x+6)}$

Solution: $\frac{x+1}{(x-5)(x+6)} = \frac{A}{x-5} + \frac{B}{x+6}$

(b) $\frac{x+1}{x(x^2+4)}$

Solution: $\frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

(c) $\frac{x+1}{x^3(x^2+4)}$

Solution: $\frac{x+1}{x^3(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$

(d) $\frac{3x^2+x+1}{(x-1)^2(x+1)^2}$

Solution: $\frac{3x^2+x+1}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$

(e) $\frac{3x^2+x+1}{x(x-1)(x^2+9)(x^2+1)}$

Solution: $\frac{3x^2+x+1}{x(x-1)(x^2+9)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2+9}$

6. Find the system of equations to solve.

(a) $\frac{x+1}{(x-5)(x+6)} = \frac{A}{x-5} + \frac{B}{x+6}$

Solution: $6A-5B=1$, $A+B=1$

(b) $\frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

Solution: $A+B=0$, $C=1$, $4A=1$

(c) $\frac{x+1}{x^3(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$

Solution: $A+D=0$, $B+E=0$, $4A+C=0$, $4B=1$, $4C=1$

(d) $\frac{3x^2+x+1}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$

Solution: $A+C=0$, $A+B-C+D=3$, $-A+2B-C-2D=1$, $-A+B+C+D=1$

(e) $\frac{3x^2+x+1}{x(x-1)(x^2+9)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2+9}$

Solution: $A+B+C+E=0$, $-A-C+D-E+F=0$, $10A+10B+9C-D+E-F=0$, $-10A-9C+9D-E+F=3$, $9A+9B-9D-F=1$, $-9A=1$

7. Solve the system of equations

   (a) $6A - 5B = 1$, $A + B = 1$
   Solution: $A = 6/11$, $B = 5/11$

   (b) $A + B = 0$, $C = 1$, $4A = 1$
   Solution: $A = 1/4$, $B = -1/4$, $C = 1$

   (c) $A + D = 0$, $B + E = 0$, $4A + C = 0$, $4B = 1$, $4C = 1$
   Solution: $A = -1/16$, $B = 1/4$, $C = 1/4$, $D = 1/16$, $E = -1/4$

   (d) $A + C = 0$, $A + B - C + D = 3$, $-A + 2B - C - 2D = 1$, $-A + B + C + D = 1$
   Solution: $A = 1/2$, $B = 5/4$, $C = -1/2$, $D = 3/4$

   (e) $A + B + C + E = 0$, $-A - C + D - E + F = 0$, $10A + 10B + 9C - D + E - F = 0$, $-10A - 9C + 9D - E + F = 3$, $9A + 9B - 9D - F = 1$, $-9A = 1$
   Solution: $A = -1/9$, $B = 1/4$, $C = -3/16$, $D = 1/16$, $E = 7/144$, $F = -5/16$

8. Integrate

   (a) $\int \left(6x^2 - 3x + 11 + \frac{6}{11(x-5)} + \frac{5}{11(x+6)}\right) dx$
   Solution: $\int \left(6x^2 - 3x + 11 + \frac{6}{11(x-5)} + \frac{5}{11(x+6)}\right) dx = 2x^3 - \frac{3}{2}x^2 + 11x + \frac{6}{11}\ln|x-5| + \frac{5}{11}\ln|x+6| + C$

   (b) $\int \left(3x^2 - 2x + 1 + \frac{1}{4x} - \frac{x}{4(x^2+4)} + \frac{1}{x^2+4}\right) dx$
   Solution: $\int \left(3x^2 - 2x + 1 + \frac{1}{4x} - \frac{x}{4(x^2+4)} + \frac{1}{x^2+4}\right) dx = x^3 - x^2 + x + \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) + \frac{1}{2}\arctan(x/2) + C$

   (c) $\int \left(x^3 - 1 - \frac{1}{16x} + \frac{1}{4x^2} + \frac{1}{4x^3} + \frac{x}{16(x^2+4)} - \frac{1}{4(x^2+4)}\right) dx$
   Solution: $\int \left(x^3 - 1 - \frac{1}{16x} + \frac{1}{4x^2} + \frac{1}{4x^3} + \frac{x}{16(x^2+4)} - \frac{1}{4(x^2+4)}\right) dx = \frac{1}{4}x^4 - x - \frac{1}{16}\ln|x| - \frac{1}{4x} - \frac{1}{8x^2} + \frac{1}{32}\ln(x^2+4) - \frac{1}{8}\arctan(x/2) + C$

   (d) $\int \left(2x + \frac{1}{2(x-1)} + \frac{5}{4(x-1)^2} - \frac{1}{2(x+1)} + \frac{3}{4(x+1)^2}\right) dx$
   Solution: $\int \left(2x + \frac{1}{2(x-1)} + \frac{5}{4(x-1)^2} - \frac{1}{2(x+1)} + \frac{3}{4(x+1)^2}\right) dx = x^2 + \frac{1}{2}\ln|x-1| - \frac{5}{4(x-1)} - \frac{1}{2}\ln|x+1| - \frac{3}{4(x+1)} + C$

   (e) $\int \left(2 - \frac{1}{9x} + \frac{1}{4(x-1)} - \frac{3x}{16(x^2+1)} + \frac{1}{16(x^2+1)} + \frac{7x}{144(x^2+9)} - \frac{5}{16(x^2+9)}\right) dx$
   Solution: $\int \left(2 - \frac{1}{9x} + \frac{1}{4(x-1)} - \frac{3x}{16(x^2+1)} + \frac{1}{16(x^2+1)} + \frac{7x}{144(x^2+9)} - \frac{5}{16(x^2+9)}\right) dx$
   $= 2x - \frac{1}{9}\ln|x| + \frac{1}{4}\ln|x-1| - \frac{3}{32}\ln(x^2+1) - \frac{1}{16}\arctan x + \frac{7}{288}\ln(x^2+9) - \frac{15}{155}\arctan(x/3) + C$

9. Determine the methods to integrate the following (simplify integrand, if necessary).

   (a) $\displaystyle\int e^{x+e^x}\, dx$
   Solution: Substitution $u = e^x$: $= \exp(e^x) + C$

   (b) $\displaystyle\int t \sec t \tan t \, dt$
   Solution: Parts with $u = t$ and $dv = \sec t \tan t \, dt$.

   (c) $\displaystyle\int \frac{2x-3}{x^3+3x}\, dx$
   Solution: Partial fractions: $= -\ln|x| + \frac{1}{2}\ln(x^2+3) + \frac{2}{\sqrt{3}}\arctan(x/\sqrt{3}) + C$

   (d) $\displaystyle\int \frac{\ln(\tan x)}{\sin x \cos x}\, dx$
   Solution: Substitution $u = \ln(\tan x)$: $= \frac{1}{2}\left(\ln|\tan x|\right)^2 + C$

   (e) $\displaystyle\int \frac{1+\tan x}{\sec x - 1}\, dx$

Solution: "Rationalize" the denominator.

$$\begin{aligned}\int \frac{1+\tan x}{\sec x - 1}\,dx &= \int \frac{1+\frac{\sin x}{\cos x}}{\frac{1}{\cos x}-1}\,dx \\ &= \int \frac{\cos x + \sin x}{1-\cos x}\,dx \\ &= \int \frac{\cos x + \cos^2 x + \sin x + \sin x \cos x}{\sin^2 x}\,dx \\ &= \int \left(\frac{\cos x}{\sin^2 x} + \cot^2 x + \csc x + \cot x\right)\,dx\end{aligned}$$

These are all straightforward. Note $\cot^2 x = \csc^2 x - 1$, which is also straightforward.

(f) $\displaystyle\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx$

Solution: Complete square and then trig substitution.

(g) $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

Solution: Long division

(h) $\displaystyle\int \frac{\cos(\frac{1}{x})}{x^3}\,dx$

Solution: Let $u = 1/x$ and convert to $\int -u\cos u\,dx$

(i) $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

Solution: $= \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx$, and then let $u = \cos x$.

(j) $\displaystyle\int \frac{\tan^{-1}(x)}{x^2}\,dx$

Solution: try parts with $u = \arctan(x)$ and $dv = \frac{1}{x^2}\,dx$.