Math 132 - September 1, 2017

Solutions

5.2: Definite Integrals

- Definition of the definite integral using Riemann Sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

- Some properties of definite integrals:

1. $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
3. $\int_a^b f(x);dx = -\int_b^a f(x)\,dx$
4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Warm-up Problems

1. **Clicker** Suppose you know nothing about $f(x)$ except the following:
$\int_1^7 f(x)\,dx = 3$ and $\int_7^9 f(x)\,dx = -1$.
What is $\int_1^9 f(x)\,dx$?
   (a) -1
   (b) 0
   (c) 2 **Correct**
   (d) 3
   (e) Impossible to determine

2. Chain Rule Practice. Remember that $[f(g(x))]' = f'(g(x))\,g'(x)$
Suppose you have a function $F(x)$ and you know $F'(x) = \sin(x^2+1)$. Find
   (a) $[F(x)]' = \sin(x^2+1)$
   (b) $\left[F(x^2)\right]' = \sin(x^4+1)\cdot 2x$
   (c) $[F(\sin x)]' = \sin(\sin^2 x+1)\cdot\cos x$
   (d) $[F(\ln x)]' = \sin((\ln x)^2+1)\cdot\frac{1}{x}$

Class Problems

Lecture Notes: Properties of the Definite Integral

The key to understanding many of these is to draw them out as areas.

0. Signed area (for $f(x)\le 0$)

I. $\int_a^b c\,dx = c(b-a)$

II. $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

III. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

IV. $\int_a^b f(x);dx = -\int_b^a f(x)\,dx$

V. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

VI. If $f(x) \geq 0$ then $\int_a^b f(x)\,dx \geq 0$

VII. If $f(x) \geq g(x)$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

VIII. If $m \leq f(x) \leq M$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

3. **Clicker** Suppose you know $\int_0^1 f(x)\,dx = 2$, $\int_1^2 f(x)\,dx = 3$, $\int_0^1 g(x)\,dx = -1$, $\int_0^2 g(x)\,dx = 4$.

   Find $\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx$

   (a) 0 **Correct**
   (b) 5
   (c) 10
   (d) ∞
   (e) Impossible to determine

   Solution: Answer is $5 + (-5) = 0$
   $\int_0^1 f(x)\,dx = 2$
   $\int_1^2 f(x)\,dx = 3$
   $\int_0^2 f(x)\,dx = 5$
   $\int_0^1 g(x)\,dx = -1$
   $\int_1^2 g(x)\,dx = 5$
   $\int_0^2 g(x)\,dx = 4$

4. Same set up as Problem 3. Find the following:

   (a) $\int_1^2 g(x)\,dx = 5$

   (b) $\int_0^2 [2f(x) - 3g(x)]\,dx = 2(5) - 3(4) = -2$

   (c) $\int_1^1 g(x)\,dx = 0$

   (d) $\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx = 3 + (-4) = -1$

5. Suppose $\int_1^b f(x)\,dx = 1 - \frac{1}{b}$. Find the following:

   (a) $\int_1^5 f(x)\,dx = \frac{4}{5}$

   (b) $\int_{1/2}^1 f(x)\,dx = 1$

   (c) $\int_1^6 3f(x) - 4\,dx = -17.5$

   (d) $\int_3^5 f(x)\,dx = -\frac{2}{15}$

Lecture Notes: Back to Riemann Sums! Remember the definite integral is defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Conceptually, we still want to think about this as "signed area under the curve", but this is the definition.

6. For this problem, we're going to compute $\int_1^3 x^2\,dx$ using a Riemann Sum.

   (a) $a = 1$

   (b) $b = 3$

   (c) $\Delta x = \frac{2}{n}$

   (d) $x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$

   (e) $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(1+2i/n)(2/n)$

   (f) Simplify the Riemann sum (goal: have an algebraic expression without any $\sum$ symbols)

$$\begin{aligned} R_n =& \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(1+2i/n)(2/n) \\ =& \sum_{i=1}^{n} (1+2i/n)^2(2/n) = \sum_{i=1}^{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)(2/n) \\ =& \sum_{i=1}^{n} \frac{2}{n} + \sum_{i=1}^{n} \frac{8i}{n^2} + \sum_{i=1}^{n} \frac{8i^2}{n^3} \\ =& \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ =& \frac{2}{n}(n) + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \end{aligned}$$

   (g) The definite integral is the limit of your Riemann sums as $n \to \infty$. Find this.

$$\begin{aligned} \lim_{n\to\infty} R_n =& \lim_{n\to\infty} \frac{2}{n}(n) + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\ =& 2 + 4 + \frac{16}{6} = \frac{26}{3} \end{aligned}$$

7. Repeat problem 6, but compute the following this time:

   (a) $\int_0^4 x^3\,dx$

   Solution: $\Delta x = 4/n$, $x_i = 4i/n$.

$$\begin{aligned} R_n =& \sum_{i=1}^{n} 64\frac{i^3}{n^3}\frac{4}{n} = \frac{256}{n^4}\frac{n^2(n+1)^2}{4} \\ \lim_{n\to\infty} R_n =& 64 \end{aligned}$$

   (b) $\int_2^5 (8x - x^2)\,dx$

   Solution: $\Delta x = 3/n$, $x_i = 2 + 3k/n$.

$$\begin{aligned} R_n =& \sum_{i=1}^{n} \left(-\frac{27i^2}{n^3} + \frac{36i}{n^2} + \frac{36}{n}\right) = -\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{36}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n \\ \lim_{n\to\infty} R_n =& 45 \end{aligned}$$

Lecture Notes: The following might be useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$